3-Dimensional Lattice Paths

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Abstract

In the first lecture of the class, we read the first section in George Polya and Robert Tarjan's *Notes on Introductory Combinatorics* , which provided a motivation for Pascal's Triangle and the subject of combinatorics. This motivation was to count the number of paths from (0,0) to (a,b) by only using the steps (0,1) and (1,0). We saw how for any point x(a,b), the number of paths from (0,0) to x was equal to the number of paths from (0,0) to (a-1,b) plus the number of paths from (0,0) to (a,b-1), which we conjectured and proved was equivalent to

$$\frac{(a+b)!}{(a!)(b!)}. \tag{1}$$

After, we were then easily able to conjecture and prove the analogous formula for the number of three-dimensional lattice paths using the steps (1,0,0), (0,1,0), and (0,0,1) was

$$\frac{(a+b+c)!}{(a!)(b!)(c!)}. \tag{2}$$

The primary purpose of our project was to create a database enumerating lattice walks in the three-dimensional lattice for various lists of atomic steps, of which could contain any number of atomic steps and any values in the steps themselves, with the exception of (0,0,0). Throughout the duration of the project, we found/examined both integer sequences that were already in the *Online Encyclopedia of Integer Sequences* (OEIS), those that were not, and those that were in the OEIS but did not have our description of lattice walks in the three-dimensional lattice using some list of atomic steps.

The secondary purpose of our project was to submit and publish interesting sequences that we found into the OEIS that were not yet published, and submit comments to existing sequences that offered our description of the sequence which were not included previously, as well as comments about interesting behaviors exhibited by a particular sequence. This also led to conjectures about how particular sequences change when the list of atomic steps is changed in a particular way, and examinations of statistical properties of particular lists of atomic steps/sequences. Our publications and most interesting conjecture are summarized below.

Published Contributions to the OEIS

As of December 14, 2020, our group has three contributions to the OEIS. One is a brand-new sequence, which is A33990, and the other two are comments to existing sequences, which are our description of the particular sequence which was not mentioned previously. These comments were made on A2898 and A208425.

The Zero Insertion Postulate

If $[x_1, x_2, x_3, x_4, x_5, ...]$ is some integer sequence corresponding to some finite list of atomic steps $\{[i_1], [i_2], ..., [i_n]\}$, where x_k represents the number of paths from (0,0,0) to (k,k,k), then the integer sequence corresponding to the list of atomic steps $\{[i'_1], [i'_2], ..., [i'_n]\}$, where each i'_k is equal to i_k with 1 added to every value in i, is $[0, x_1, 0, x_2, 0, x_3, 0, x_4, 0, x_5, 0, ...]$

e.g. Steps: (0,1,2),(0,2,1),(1,0,2),(1,2,0),(2,0,1),(2,1,0) –> (1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)
Sequence: [0, 6, 12, 90, 360, 2040, 10080,...] –> [0, 0, 0, 6, 0, 12, 0, 90, 0, 360, 0, 2040, 0, 10080, 0,...]